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The effective mass of a point mass moving along a string on a Winkler foundation☆

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Abstract

A simple in form and physically clear asymptotic solution of the problem of the motion (without friction) of a point mass acted upon by a specified external force on a string on a Winkler foundation is obtained, taking into account the wave drag on the motion. It is shown that the point mass moves along the string in the same way as a point with a variable velocity-dependent mass would move when acted upon solely by an external force (ignoring drag).

The problem of the motion of a point mass along a string under the action of a specified external force, when the longitudinal component of the force of interaction between the point and the string is a so-called vibrational pressure[1–3] (the wave drag to the motion), was considered previously in Refs. 4–7 in different formulations. However, for subcritical motion only particular results were obtained, and in an unwieldy form, due to the need to investigate a non-linear non-stationary problem for a partial differential equation. Note that problems with a similar mathematical formulation may arise in the theory of phase transitions in elastic solids (in the low-strain approximation) and in the theory of dislocations.[8,9]

Below we obtain an asymptotic solution of the problem in a simple form with zero initial conditions, which allows of a clear physical interpretation. The problem can also be simplified considerably in the case of non-zero initial conditions. The solution is based on the results of an asymptotic analysis of the motion of this system for a specified law of motion of the point mass;[10] generalizations of these results are obtained. Among the publications on this theme we should also mention investigations of simpler special cases of the problem,[11,12] considered in Ref. 10.

We separately consider the motion of a point mass along a string without an elastic foundation. It is shown that the wave drag in this case behaves like a viscous-friction force.

1. Formulation of the problem

Consider a tight string, rectilinear in the undeformed state, on an elastic Winkler foundation. A point mass, to which a specified external force is applied, moves without friction along the string.

We will introduce the following notation: $\mathbf{e}_{||}$ and $\mathbf{e}_{\perp}$ are unit vectors related to the direction of the string and to a certain direction perpendicular to it, respectively, $u(x, t)\mathbf{e}_{\perp}$ is the transverse displacement of a point on the string with

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coordinate x at the instant of time t, T_0 is the tension in the unperturbed string, ρ is the density of the string per unit length, $c = \sqrt{T_0/\rho}$ is the velocity of sound (the critical velocity), $l(t)$ is the coordinate of the point mass, $k_0 = T_0 k > 0$ is the coefficient of elasticity of the foundation, $\mathbf{f}_0 = T_0(f_{||}\mathbf{e}_{||} + f_{\perp}\mathbf{e}_{\perp})$ is the specified external force acting on the point mass, $\mathbf{F}_0 = T_0(F_{||}\mathbf{e}_{||} + F_{\perp}\mathbf{e}_{\perp})$ is the force acting on the string from the side of the point mass, and $m_0 = T_0 m$ is the mass of the point mass. The jump and mean value of the arbitrary quantity ζ, discontinuous at $x = l(t)$, will be denoted by [ζ] and ⟨ζ⟩ respectively.

We will consider the subcritical motion of the point mass along the string ($|\dot{l}| < c$ for all t). Transverse displacements of points on the string must satisfy the equation

$$u'' - c^{-2}\ddot{u} - ku = -F_{\perp}\delta(x - l(t)) \tag{1.1}$$

The equations of transverse oscillations of the point mass and its longitudinal motion along the string have the form

$$md^2u(l(t), t)/dt^2 = f_{\perp} - F_{\perp}, \quad m\ddot{l} = f_{||} - F_{||} \tag{1.2}$$

The longitudinal force ($-F_{||}$), acting on the point mass from the string side, is the wave drag to the motion[1–3,8,13]

$$-F_{||} = -\kappa^2[u'^2]/2, \quad \kappa = \kappa(t) = \sqrt{1 - (\dot{l}(t))^2/c^2} \tag{1.3}$$

The nature of this force caused some discussion, cf. in Refs. 8,13,14 and will not be commented on here.

We will consider arbitrary fairly smooth initial conditions for the displacements u, which are non-zero only in a compact subset of the real axis. We will correspondingly require that $u \equiv 0$ for all fairly large $|x|$ for any finite t. We will assume that

$$l(0) = 0, \quad \dot{l}(0) = \upsilon_0, \quad |\upsilon_0| < c \tag{1.4}$$

We will put

$$f_{\perp} = O(\varepsilon), \quad f_{||} = O(\varepsilon^2) \tag{1.5}$$

(ε is a formal small parameter). We will assume that the forces $f_{\perp}, f_{||}$ are (non-strict) monotonic slowly varying functions of time

$$\dot{f}_{\perp} = O(\varepsilon^2), \quad \dot{f}_{||} = O(\varepsilon^2) \tag{1.6}$$

It was shown in Ref. 13 that the formulation of the problem presented above for Eqs. (1.1) and (1.2) follows from the complete geometrically non-linear formulation of the problem for conditions (1.5) and is only correct when these conditions are satisfied. If the longitudinal force $f_{||}$ is so large that the second relation of (1.5) is not satisfied, then instead of Eq. (1.1) we must consider a system of two coupled equations, describing the transverse and longitudinal motions of the string, respectively. When conditions (1.5) are satisfied, the longitudinal dynamics of the string can be considered independently after solving the problem for Eqs. (1.1) and (1.2).

The purpose of this research was to determine the law of motion $l(t)$ of the point mass. The problem is non-stationary (the motion occurs with a variable velocity) and non-linear (by virtue of the second equation of (1.2)).

The solution of Eq. (1.1) can be represent as

$$u = u_{\text{ext}} + u_{\text{int}} \tag{1.7}$$

where u_{ext} is the contribution from the non-zero initial conditions and u_{int} is the contribution from the right-hand side of Eq. (1.1), which satisfies the zero initial conditions. It follows from Eq. (1.1) that

$$-F_{\perp} = \kappa^2[u'] \tag{1.8}$$

Hence the second equation of (1.2) can be rewritten in the equivalent form

$$m\ddot{l} = F_{\perp}\langle u'\rangle + f_{||} \tag{1.9}$$

If the initial data is sufficiently smooth, the solution u_{ext} is also smooth ($[u'_{\text{ext}}] = 0$). Consequently

$$m\ddot{l} = \Phi_{\text{int}} + \Phi_{\text{ext}} + f_{\parallel}; \quad \Phi_{\text{int}} = F_{\perp}\langle u'_{\text{int}}\rangle, \quad \Phi_{\text{ext}} = F_{\perp}u'_{\text{ext}} \tag{1.10}$$

The term Φ_{int} on the right-hand side of Eq. (1.10) corresponds to self-action (interaction between the point mass and the field which it generates), and Φ_{ext} corresponds to interaction with the external field. Using expression (1.8) it can be shown that

$$\Phi_{\text{int}} = -\kappa^2[u'^2_{\text{int}}]/2 \tag{1.11}$$

Note that $\Phi_{\text{int}} \equiv 0$ if $\dot{l} = 0$ for all $t > 0$, since $u'^2_{\text{int}} \equiv 0$ by virtue of the symmetry of the solution of Eq. (1.1) with zero initial conditions.

Below, we shall investigate the effect of self-action, i.e. we shall investigate the effect of the wave drag Φ_{int} on the longitudinal motion of the point mass.

2. The solution of the problem for a specified law of motion of the point mass

In this section we will assume that $u_{\text{ext}} = 0$, so that $u \equiv u_{\text{int}}$. To solve the problem of the motion of a point mass acted upon by a specified external force we will use the results obtained in Ref. 10 for the case of a specified law of motion $l(t)$ of the point mass along the string. The asymptotic solution was constructed by assuming that the function $\dot{l}(t)$ is (non-strictly) monotonic and that the following requirements are jointly satisfied

$$\ddot{l} = O(\varepsilon^2), \quad t = O(\varepsilon^{-2}), \quad |\dot{l}| = \left|\int_0^t \ddot{l}(\zeta)d\zeta\right| < c \tag{2.1}$$

We have used the approach based on asymptotic stationary-phase and multiscale methods. Below we will briefly present the fundamental formulae obtained previously in Ref. 10, and we will also give some necessary generalizations of these formulae. Note that the small parameter ε introduced in Ref. 10 corresponds to the quantity ε^2 in the notation used in the present paper.

Consider a system of coordinates $\xi = x - l(t)$, $\tau = t$, which moves together with the point mass. The solution of the problem $u(\xi, \tau)|_{\xi=0}$ is approximated by the sum of oscillation modes $w^{\Omega}_{\delta}(0, \tau)$ with frequency $\delta\Omega$, $\delta = \pm 1$, $\Omega \in \sigma$, where σ is a set of non-negative frequencies (possibly slowly varying), characteristic for this system. In this case there are two such systems: a slowly varying frequency of naturally localized oscillations $\Omega_0 = \Omega_0(\tau)$, where

$$\Omega_0^2 = 2\frac{\sqrt{1 + m^2c^4\kappa^2k} - 1}{m^2c^2} \tag{2.2}$$

and zero frequency, corresponding to the contribution from the external slowly varying force $f_{\perp}$, i.e. $\sigma = \{\Omega_0, 0\}$.

The oscillation amplitudes are also slowly varying functions.

Under conditions (2.1) the vertical motion of the point mass is described by the relation[10]

$$w(0, \tau) = \frac{C_0 c\kappa}{\sqrt{\Omega_0(m^2c^2\Omega_0^2 + 2)}}\cos\left(\int_0^\tau \Omega_0(\zeta)d\zeta - D_0\right) + \frac{f_{\perp}}{2\kappa\sqrt{k}} + O(\varepsilon^2) \tag{2.3}$$

In the special case when $f_{\perp}(t) = f_0 H(t)$ [10], where $H(t)$ is the Heaviside function, we have

$$C_0 = f_0 Z, \quad Z = -\frac{mc\sqrt{\Omega_0(0)}}{\kappa(0)\sqrt{(m^2c^2\Omega_0^2(0) + 2)}}, \quad D_0 = 0 \tag{2.4}$$

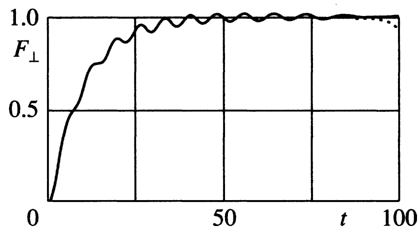

Fig. 1.

In the more general case we will assume that the slowly varying function $f_\perp(t)$ is equal to zero when $t<0$, is bounded and has the Fourier transform

$$f_\perp^F(\Omega) = \int_{-\infty}^{+\infty} f_\perp(t)\exp(i\Omega t)dt \tag{2.5}$$

(understood, if necessary, in a generalized sense). If $f_\perp^F(\Omega)$ has no singularities in a certain neighbourhood of each point on the real axis, with the exception, possibly, of the zero point, then, proceeding in the same way as described in Ref. 10, we obtain

$$C_0 = |\psi|Z, \quad D_0 = \arg\psi, \quad \psi = -if_\perp^F(\Omega_0(0))\Omega_0(0) \tag{2.6}$$

For example, if

$$f_\perp(t) = f_0H(t)(1-\exp(-\lambda t)) \tag{2.7}$$

where $f_0>0$, $\lambda>0$, we have[15]

$$\psi = \frac{i\lambda f_0}{\Omega_0(0)+i\lambda}; \quad |\psi| = \frac{\lambda f_0}{\sqrt{\Omega_0(0)^2+\lambda^2}}, \quad \arg\psi = \frac{\pi}{2} - \mathrm{arctg}\frac{\lambda}{\Omega_0(0)} \tag{2.8}$$

Numerical calculations, carried out for the case (2.7), using the same method as in Ref. 10, completely confirm the correctness of formulae (2.3) and (2.6) (see Fig. 1). The dashed curve represents the numerical results for the unknown function $F_\perp(t)$, found by simultaneous solution of Eq. (1.1) and the first equation of (1.2) with zero initial conditions, while the continuous curve represents the analytical results. The calculations were carried out for the following values of the parameters

$$c=1, \quad k=1, \quad m=1, \quad \ddot{l} = \text{const} = 0.01, \quad f_0=1, \quad \gamma=0.1, \quad f_\perp=1, \quad v_0=0$$

It can be seen that the disagreement between the graphs of the asymptotic and numerical solutions becomes appreciable only as the velocity $\dot{l}$ of the point mass approaches the critical value c (when $t=100$).

For $\xi\neq 0$ the solution will be sought in the form[10]

$$\begin{gathered} w(\xi,\tau) = \sum_{\Omega(0,T)\in\sigma}\sum_{\delta=\pm1} w_\delta^\Omega(\xi,\tau) \equiv \sum_{\Omega(0,T)\in\sigma}\sum_{\delta=\pm1} W_\delta^\Omega(X,T)\exp\varphi_\delta^\Omega(\xi,\tau) \\ X = \varepsilon^2\xi, \quad T = \varepsilon^2\tau, \quad (\varphi_\delta^\Omega)'_\xi = i\omega_\delta^\Omega(X,T), \quad (\varphi_\delta^\Omega)'_\tau = -i\delta\Omega(X,T) \end{gathered} \tag{2.9}$$

Hence, the solution when $\xi\neq 0$ is approximated by the finite sum $w(\xi,\tau)$ of the oscillation modes w_δ^Ω with slowly varying frequencies $\delta\Omega(X,T)$, $\delta=\pm1$, such that the functions $\Omega(X,T)$ are extensions on $X\neq 0$ of the frequencies $\Omega(T)$ from the set σ. Here the wave numbers $\omega_\delta^\Omega(X,T)$ (generally speaking, complex) are defined by virtue of the dispersion relation

$$\omega^2 - 2B(\delta\Omega)\omega + A^2(\delta\Omega) = 0 \tag{2.10}$$

and satisfy the equation

$$\delta\Omega'_X + \omega'_T = 0 \tag{2.11}$$

which follows from the last two equations of (2.9). Here and henceforth the superscript Ω and the subscript δ on the wave numbers ω are omitted for brevity. Eq. (2.10) and the boundedness condition give, as $\xi \to \infty$

$$\omega = B(\delta\Omega) + i\,\mathrm{sign}\,\xi S(\delta\Omega) \tag{2.12}$$

We have used the following notation in Eqs. (2.10) and (2.12)

$$A^2(\Omega) = \kappa^{-2}(k - \Omega^2/c^2), \quad B(\Omega) = \kappa^{-2}c^{-2}\dot{l}\Omega, \quad S^2(\Omega) = A^2(\Omega) - B^2(\Omega) \tag{2.13}$$

Here $A(\Omega) > 0$, $S(\Omega) > 0$ if $\Omega^2 < kc^2\kappa^2$. The function w and the derivative w'_τ must satisfy the conditions of continuity when $\xi \to \pm 0$. Here the derivatives W'_X and φ'_ξ, generally speaking, are discontinuous functions when $\xi = 0$.

It follows from Eq. (2.3) that

$$W_\delta^{\Omega_0}(0, T) = C_0 \frac{c\kappa}{2\sqrt{\Omega_0(m^2c^2\Omega_0^2 + 2)}}, \quad W_\delta^0(0, T) = \frac{f_\perp}{4\kappa\sqrt{k}} \tag{2.14}$$

Using the representation for the solution (2.9), (2.10) and employing the multiscale method, we can obtain the following expression for $(W_\delta^\Omega)'_X$ when $X = \pm 0$ (see Ref. 10)

$$W'_X\big|_{X=\pm 0} = -\frac{\mathrm{sign}\,X}{2iS(\delta\Omega)c^2\kappa^2}(F_0 + F_1 + F_2) + o(\varepsilon^2)$$
$$F_0 = \ddot{l}\omega W, \quad F_1 = 2(\dot{l}\omega + \delta\Omega)W'_T, \quad F_2 = (-c^2\kappa^2\omega'_{\delta\Omega}\omega'_T + 2\dot{l}\omega'_T + \delta\Omega'_T)W \tag{2.15}$$

In formulae (2.15) the superscript Ω and the subscript δ on the function W are omitted for brevity.

Note that relations (2.15) were obtained previously[10] only for the amplitude W^{Ω_0} of the localized mode of oscillations. Their derivation for the amplitude W^0 of the mode with zero frequency is analogous.

We will calculate the wave drag (1.3), using the representation for the solution (2.9). When $\xi = \pm 0$ we have

$$u'^2 \simeq 2(w_1^{\Omega_0})'_\xi(w_{-1}^{\Omega_0})'_\xi + 4(w_{\pm 1}^0)'^2_\xi + \Psi$$
$$\Psi = (w_1^{\Omega_0})'^2_\xi + (w_{-1}^{\Omega_0})'^2_\xi + 4(w_{\pm 1}^0)'_\xi(w_{-1}^{\Omega_0})'_\xi + 4(w_{\pm 1}^0)'_\xi(w_1^{\Omega_0})'_\xi \tag{2.16}$$

The function Ψ is an oscillating function of the fast phase φ^{Ω_0} with zero mean. Hence, the average contribution from it over a period of the natural oscillations is equal to zero. Differentiating the function w_δ^Ω with respect to ξ, taking into account representations (2.9), we obtain

$$[(w_1^\Omega)'_\xi(w_{-1}^\Omega)'_\xi] = i\sum_{\delta = \pm 1} W_\delta[(W_{-\delta}^\Omega)'_\xi\omega_\delta] + o(\varepsilon^2) \tag{2.17}$$

Here we have taken into account the fact that $[\omega_{-1}\omega_1 = 0]$, by virtue of relations (2.12) and (2.13).

We will now consider the steady motion of a point mass along the string with a subcritical constant velocity $|\dot{l}| = \mathrm{const} < c$. Suppose also that $f_\perp = \mathrm{const}$. In this case the amplitudes of the modes $W_\delta^{\Omega_0}$ and W_δ^0 are constant. It now follows from relations (1.3), (2.16) and (2.17) that the wave drag, on average, over a period of the natural oscillations is equal to zero. If there are no localized oscillations (the first term on the right-hand side of expression (2.3) is equal to zero), the drag is exactly equal to zero. This effect is a consequence of the symmetry of the profile of the string in a mobile system of coordinates when there is a Winkler foundation. At the same time, in the case of accelerated motion of the point mass $\ddot{l} \neq 0$ the profile of the string becomes asymmetrical, which leads to the appearance of wave drag.

3. The solution of the problem for a specified external force

We will use the results obtained for the case of a specified law of motion $l(t)$ of the point mass. Namely, we will express the vibrational pressure force Φ_{int} (1.11) in terms of $\ddot{l}(t)$ using formulae (2.3), (2.9), (2.12) and (2.15)–(2.17).

Below we will consider the case when the first term on the right-hand side of relation (2.3), which describes the natural localized oscillations, can be neglected compared with the second term (and, as a consequence, we can put $F_\perp \simeq f_\perp$). This is possible in two cases:

1) when $C_0 \simeq 0$; in particular, this condition will be satisfied (see formula (2.6)) for a fairly small mass of the point mass m or, if the force $f_\perp$ is specified by formula (2.7)) and the coefficient λ is fairly small;
2) for a velocity $\dot{l}(t)$ sufficiently close to c (provided that the acceleration $\ddot{l}$ is fairly small, so that the solution can be assumed to be non-resonant); in this case we obtain $W_\delta^{\Omega_0}(0, T) \simeq 0$ by virtue of the first formula of (2.14) (see Ref. 10).

By virtue of relation (2.16) when $\xi = \pm 0$ we have

$$u_{\text{int}}'^2 \simeq w_\xi'^2 = 4(w_{\pm 1}^0)_\xi'^2 \tag{3.1}$$

Using the results obtained together with Eq. (1.11), we find

$$\Phi_{\text{int}} = -2\kappa^2[(w_{\pm 1}^0)_\xi'^2] \tag{3.2}$$

Substituting expressions (2.12) and (2.15) into Eq. (2.17) and dropping small higher-order terms, we obtain (the superscript $\Omega = 0$ and the subscript $\delta = \pm 1$ on the function W are omitted)

$$[(w_{\pm 1}^0)_\xi'^2] \simeq \frac{W}{S(0)c^2\kappa^2}(\dot{l}(A^2(0))_T' W + 2\ddot{l}A^2(0)W + 4\dot{l}A^2(0)W_T') \tag{3.3}$$

Substituting the estimate (3.3) (where W is expressed by the second formula of (2.14)) into (3.2), we obtain, after simplification,

$$\Phi_{\text{int}} = -m\alpha\left(\frac{3\dot{l}^2}{c^5\kappa^5} + \frac{1}{c^3\kappa^3}\right)\ddot{l} - \frac{m\dot{\alpha}\dot{l}}{c^3\kappa^3}, \quad \alpha = \frac{cf_\perp^2}{4m\sqrt{k}} \tag{3.4}$$

Substituting the expression for Φ_{int} into the first equation of (1.10) and integrating, we arrive at the final result

$$(M\dot{l})^{\bullet} = F_\perp u_{\text{ext}}' + f_\parallel, \quad M = m(1 + \alpha\kappa^{-3}) \tag{3.5}$$

Hence, the effect of self-action (non-linear interaction of the point mass with the elastic field, radiated by itself) leads to the occurrence in the point mass of an "effective mass" M, which depends on the velocity $\dot{l}$. The quantity M becomes infinite as the velocity $\dot{l}$ approaches the critical value c. Note that the "additional mass" $M - m$ is independent of m and therefore occurs even if $m = 0$.

We will consider the case of non-zero initial conditions for u. The quantity $F_\perp = f_\perp$, which defines the interaction of the point mass with the fields caused by non-zero initial conditions, plays the role of an additional external longitudinal force. To solve the problem it is necessary first of all to obtain u_{ext}. To do this we must solve Eq. (1.1) by putting $F_\perp = 0$ in it with the corresponding initial conditions. At the second stage, putting $F_\perp = f_\perp$, we need to solve the first equation of (3.5) and thereby determine the law of motion $l(t)$ of the point mass. At the third stage we must solve Eq. (1.1) with zero initial conditions and determine u_{int}. Hence, the investigation reduces to solving two linear problems for a partial differential equation and solving a non-linear ordinary differential equation (the first equation of (3.5)), which is much simpler than the initial problem.

4. The motion of a point mass along a free string

We will consider separately the case $k = 0$, corresponding to a string without a Winkler foundation. The Klein-Gordon Eq. (1.1) in this case degenerates into a wave equation, which possesses quite different properties. We first note that, unlike the case $k > 0$, Eq. (1.1) when $k = 0$ does not allow of a self-similar solution for the case of the motion of a load $F_\perp$ = const with constant velocity $\dot{l}$. Bending under the load turns out to be a linearly increasing function of time t, and hence the problem of an infinite unattached string, generally speaking, is not completely correct. Moreover, the solution u_{int}, unlike the case $k > 0$, is asymmetrical about $\xi = 0$ for any velocity $\dot{l} \neq 0$. In fact, for any subcritical motion of the load $l(t)$ we have (see Ref. 16, formulae (5)–(7))

$$u'_{\text{int}} = -\frac{F_\perp(t')}{2}\left(\frac{H(x-l(t'))}{1-\dot{l}(t')/c} - \frac{H(l(t')-x)}{1+\dot{l}(t')/c}\right), \quad t' + \frac{|x-l(t')|}{c} = t \tag{4.1}$$

Bearing in mind the fact that $t' = t$ when $\xi = \pm 0$, we obtain

$$u'_{\text{int}}(t)|_{\xi=\pm 0} = \frac{\mp F_\perp(t)}{2(1\mp \dot{l}(t)/c)} \tag{4.2}$$

Whereby

$$[u'^2_{\text{int}}] = \frac{4F_\perp^2 \dot{l}}{c\kappa^4} \tag{4.3}$$

Hence, $[u'^2_{\text{int}}] \neq 0$ when $\dot{l} \neq 0$, and therefore, according to formula (1.11), a non-zero wave drag always acts on the point mass for any non-zero velocity $\xi = \pm 0$. This is related to the fact that the effective mass M in solution (3.5) becomes infinite as $k \to +0$. We recall that if the conditions for which formula (3.5) holds are satisfied and $f_\perp$ = const, then when $\dot{l}$ = const there will be no drag, and it will only occur as a reaction to the non-zero acceleration $\ddot{l}$ of the point mass.

We will obtain the law of motion of the point mass when $k = 0$. For simplicity we will confine ourselves to the case when the mass of the point mass is fairly small, so that we can put $F_\perp \simeq f_\perp$. Substituting expression (4.3) into relations (1.10) and (1.11), we have

$$m\ddot{l} + \frac{2f_\perp^2}{c\kappa^2}\dot{l} = F_\perp u'_{\text{ext}} + f_\parallel \tag{4.4}$$

The result obtained, like formula (4.2), holds for any form of motion $l(t)$, such that $|\dot{l}| < c$ for all t. Thus, when $k = 0$ for small velocities $\dot{l}$, the wave drag behaves as a viscous-friction force; at finite subcritical velocities $\dot{l}$, this friction becomes non-linear.

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